Contents

1. Introduction

This report aims to analyse a portfolio of seven selected stocks, including their historical prices, expected returns, volatility, correlations, and risk. The portfolio will be optimized for a target return using an appropriate solver function, and the efficient frontier curve will be plotted. The Sharpe ratios will be calculated using a risk-free investment, and the equation of the Capital Market Line will be determined. Linear regression analysis will be used to calculate the beta of each asset in the portfolio and to estimate the Value at Risk (VaR) of the portfolio. The volatility of a single asset in the portfolio will be estimated using ARCH/GARCH models, and the best model will be identified.

2. Portfolio management

2.1. Asset returns, volatilities and correlations

Seven different stocks from different sectors of industry have been chosen for analysis:

1. FANG - Diamondback Energy, Inc., an independent oil and natural gas company, its focus is on the acquisition, development, exploration, and exploitation of unconventional and onshore oil and gas reserves located in the Permian Basin in West Texas.
2. ATCO - Atlas Corp. operates containerships and manages assets.
3. SHEL - Shell plc a petrochemical and energy company that operates around the world.
4. AAPL - Apple Inc. a global leader in the marketing, design and manufacturing of smartphones, tablets, wearables, and accessories.
5. NGLOY - Anglo American plc an international mining company. In addition to diamonds and other metals, the company also searches for coal, metallurgical coal, and iron ore; nickel, polyhalite, and manganese ores; and alloys made from these minerals.
6. AMRK - A-Mark Precious Metals, Inc., trading company for precious metals, together with its subsidiaries.
7. TJX - The TJX Companies, Inc., in addition to its subsidiaries, the company operates as a retailer for off-price apparel and home fashions.

Calculating the returns

In order to calculate the expected returns of each asset in portfolio we need to calculate the return of each asset in the portfolio. The returns of the assets are used to represent the performance of a stock over period. This allows investors to measure how much the asset's price has changed over a given period and compare the performance of different assets [1].
We can calculate the return by taking the difference between the final price and the initial price of an asset, divided by the initial price.

$$R_i = \frac{S_{i+1} - S_i}{S_i}$$

where S_i - the value of an asset on day i,
R_i – daily return.

Or we can use log return also known as continuously compounded returns, are calculated by taking the natural logarithm of the final price divided by the initial price.

Log return = ln $(\frac{S_{i+1}}{S_i})$

For further analysis we will be using log returns of the stocks due to number of advantages that make them useful for financial analysis. For example, log returns are additive and useful for comparing returns between assets with different levels of volatility, because they are normalized by the initial price, et al [2].

And daily returns of the assets in the portfolio we can calculate using “colMeans” function which is produces the average value of the column. We can multiply it by 250 to get annualized returns of the assets.

Calculating the volatilities

To calculate the daily volatility of an asset we can use “StdDev” R function, and this is used as a proxy measure for daily volatility.
For the annual volatility or annual standard deviation, the daily volatility can be multiplied by square root of the number of trading days in a year, which is 250.

Stock names	FANG	ATCO	SHEL	AAPL	NGLOY	AMRK	TJX
Daily return	0.000339	0.000049	0.000149	0.000859	0.001303	0.000826	0.000476
Annual return	0.084741	0.012143	0.037173	0.214866	0.325821	0.206383	0.119083
Daily volatility	0.035680	0.028634	0.021479	0.019156	0.032021	0.029550	0.018381
Annual volatility	0.564144	0.452741	0.339606	0.302883	0.506293	0.467227	0.290634

Table 1. Returns and volatilities of assets in the portfolio

Correlations between the asset returns

	FANG	ATCO	SHEL	AAPL	NGLOY	AMRK	TJX
FANG	1	0.376443	0.681491	0.271444	0.407423	0.092352	0.332648
ATCO	0.376443	1	0.390008	0.227597	0.339256	0.088604	0.315839
SHEL	0.681491	0.390008	1	0.307917	0.580667	0.075908	0.416071
AAPL	0.271444	0.227597	0.307917	1	0.349563	0.123143	0.422913
NGLOY	0.407423	0.339256	0.580667	0.349563	1	0.122751	0.328555
AMRK	0.092352	0.088604	0.075908	0.123143	0.122751	1	0.12363
TJX	0.332648	0.315839	0.416071	0.422913	0.328555	0.12363	1

Table 2. Correlation matrix between asset returns

Correlation, in the finance, is a statistic that measures the degree to which two securities move in relation to each other. Correlations are used in advanced portfolio management, computed as the correlation coefficient, which has a value that must fall between -1.0 and +1.0 [3]. We can plot it using “ggcorrplot()” function:

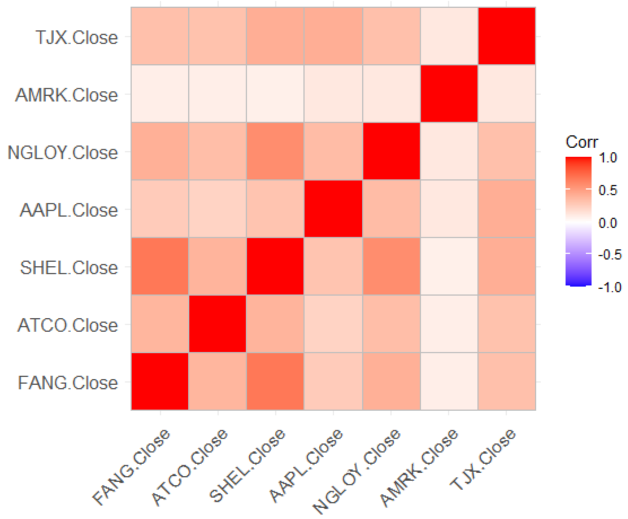

Figure 1. Correlation heatmap between stocks

2.2. Portfolio risk and portfolio optimization. Drawing the *efficient frontier curve*.

Portfolio risk and optimum asset weights in portfolio

To identify the optimum weights of the assets in the portfolio we will be using the “portfolio.spec()” and “optimize.portfolio()” functions of “PortfolioAnalytics” package in R. The “portfolio.spec()” function creates a portfolio specification object that is used to define the constraints, objectives, and other characteristics of a portfolio. In our case we have constraints which will define the maximum and minimum amount of total weight of assets. The second constraint for maximum and minimum weight of each asset in portfolio. Third constraint is for our chosen target return. Also, we have two objectives which are used to maximize the portfolio's return, and to minimize the portfolio's risk. The “optimize.portfolio()” function is then called to optimize the portfolio based on the specified constraints and objectives.

	FANG	ATCO	SHEL	AAPL	NGLOY	AMRK	TJX	Expected returns	StdDev
W1	0.050	0.050	0.050	0.354	0.282	0.163	0.050	0.2189	0.2707
W2	0.142	0.142	0.142	0.142	0.142	0.142	0.142	0.1464	0.2654
W3	0.095	-0.305	-0.837	0.827	0.724	0.308	0.187	0.4771	0.4739
W4	0.100	0.100	0.100	0.242	0.218	0.138	0.100	0.1811	0.2648
W5	1.28E-17	3.29E-17	-6.66E-17	4.53E-01	3.52E-01	1.95E-01	-2.78E-17	0.2567	0.2878

Table 3. Expected returns and standard deviations of portfolios

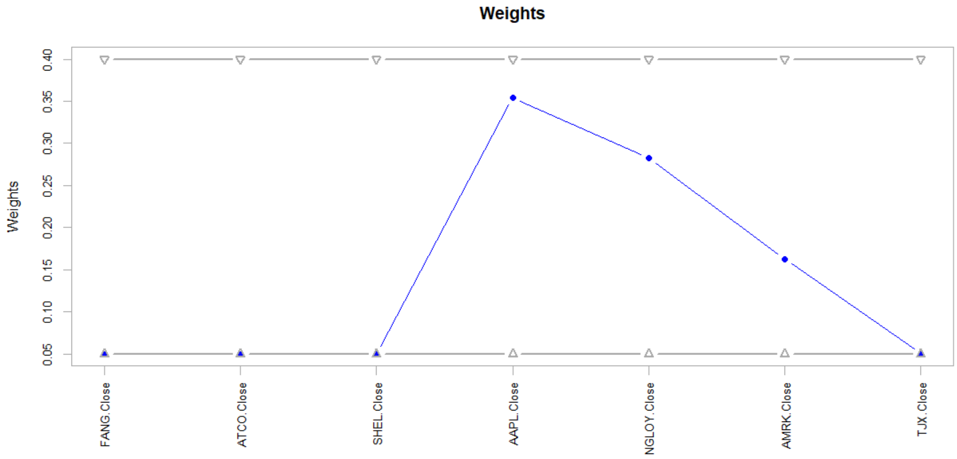

Figure 2. Optimized weights of the assets in the portfolio (W1)

The portfolio risk is a measure of the amount of risk that the portfolio is exposed to. It is a measure of the volatility of the portfolio's returns. And it is calculated with formula as below. From Table 3 we can see that all the of the portfolios, apart from W3, have risk values higher than the expected returns. W3 have a slightly higher expected return than the volatility, but the risk value is too high – 0.4739. And W1 and W5 seems to have more or less reasonable risk and return ratio.

$$\sigma_P = \sqrt{\sum_{i=1}^{N}\sum_{J=1}^{N} W_i W_j \rho_{ij} \sigma_i \sigma_i}$$

	FANG	ATCO	SHEL	AAPL	NGLOY	AMRK	TJX
FANG	0.31780186	0.09598189	0.13039556	0.04632017	0.11620589	0.02432749	0.05440228
ATCO	0.09598189	0.20463577	0.05986357	0.03116477	0.07764586	0.01871068	0.04147965
SHEL	0.13039556	0.05986357	0.11518158	0.03165449	0.09969552	0.01206626	0.04097163
AAPL	0.04632017	0.03116477	0.03165449	0.09164277	0.05353580	0.01745597	0.03715685
NGLOY	0.11620589	0.07764586	0.09969552	0.05353580	0.25594498	0.02899761	0.04822039
AMRK	0.02432749	0.01871068	0.01206626	0.01745597	0.02899761	0.21800118	0.01675992
TJX	0.05440228	0.04147965	0.04097163	0.03715685	0.04822039	0.01675992	0.08437576

Table 4. Variance-Covariance Matrix

Efficient Frontier Curve

The optimal portfolio risk is the level of risk that is most appropriate for an investor's risk tolerance and investment objectives. It is the level of risk that maximizes the expected return of the portfolio for a given level of risk or minimizes the risk for a given level of expected return.
The optimal portfolio risk can be determined using the efficient frontier curve, which is a graphical representation of the trade-off between expected return and risk for a given set of investments. An efficient frontier curve determines which portfolios provide the highest expected returns for a particular amount of risk or which portfolios have the lowest risk for a given amount of return.

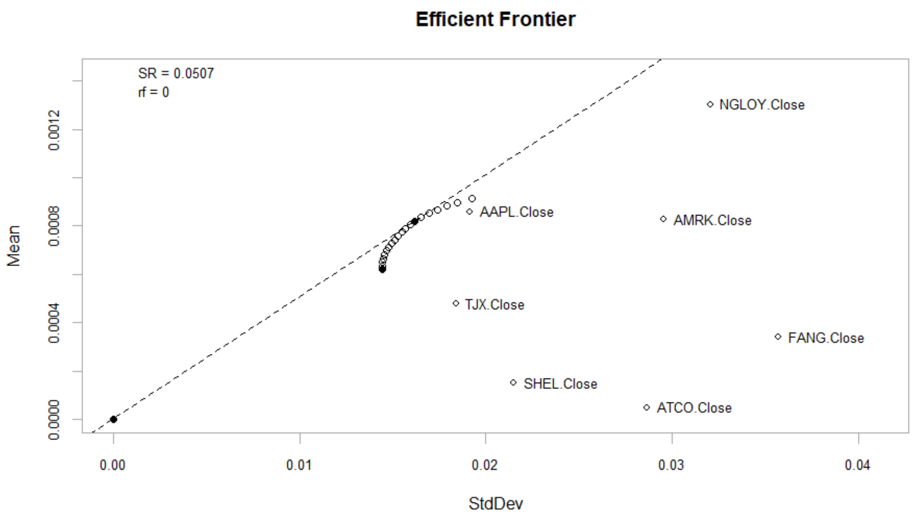

Figure 3. Efficient Frontier Curve

Only portfolios on or within the parabola are feasible [8].

2.3. Sharpe ratio and Capital Market Line

Calculate the Sharpe Ratio

The Sharpe ratio is a measure of the risk-adjusted return of an asset or a portfolio, calculated as the excess return over the risk-free rate divided by the volatility of the returns. It is used to compare the performance of different investments and to evaluate the trade-off between risk and return.
To calculates the Sharpe ratio for each combination the following formula can be used: (expected return – risk free rate)/volatility or $S\_R = \frac{E(r_p - r_f)}{\sigma_p}$.

Portfolios	W1	W2	W3	W4	W5
Sharpe Ratio	0.808539	0.551808	1.006569	0.683989	0.891778

Table 5. Sharpe ratio of the portfolios

A higher Sharpe ratio indicates a better risk-adjusted return and is generally considered an attractive investment.

Equation of the Capital Market Line

The capital market line (CML) represents portfolios that optimally combine risk and return. It is a theoretical concept that represents all the portfolios that optimally combine the risk-free rate of return and the market portfolio of risky assets [5]. And the equation is as follows:

$$R_P = R_f + \left(\frac{R_M - R_f}{\sigma_M}\right)\sigma_P$$

where:
R_P – portfolio return
R_f – risk free rate
σ_P – standard deviation of portfolio returns
R_M – market return
σ_M – standard deviation of market returns

In the R programming language we can use “Return.portfolio()” function to calculate the daily portfolio returns and annualize it using “table.AnnualizedReturns()” function. From the Table 6 below portfolio W5 seems better option from other portfolios. But none of these are considered good portfolios, due to high risk and less return ratio.

Portfolio Returns	**W1**	**W2**	**W3**	**W4**	**W5**
Annualized Return	0.1590	0.0962	0.2884	0.1326	0.1911
Annualized Std Dev	0.3064	0.2748	0.3975	0.2920	0.3226
Annualized Sharpe (Rf=1.51%)	0.4622	0.2904	0.6768	0.3957	0.5370

Table 6. Annualized portfolio returns

The CML is important because it helps investors understand the trade-off between risk and return. It shows that, in general, investors can expect to receive higher returns for taking on more risk. However, it also shows that there is a point at which the expected return begins to level off, even as risk continues to increase. This suggests that there is a limit to the amount of risk that an investor should be willing to take on in order to maximize their expected return.

2.4. Beta and Value at Risk (5%)

Calculating the Beta

Beta is a measure of the volatility or portfolio compared to the market as a whole (usually the S&P 500). Stocks with the beta results higher than 1 can be interpreted as more volatile than the market [4]. Mathematically, the beta of an asset is the ratio of the covariance between the return on the asset and the return on the market portfolio to the variance of the market portfolio:

$$\beta_i = \frac{Cov(R_i R_M)}{Var(R_M)}$$

First, we have to get the market returns and join the portfolio return with market return to one data frame. We can use the “gather()” function from the “tidyverse” package to reshape the data frame “DailyReturnsBenchmJoinedDF” so that the asset returns are in a single column. Then we group the data by asset and applies the “lm()” function to estimate a linear model of the asset returns as a function of the market returns (SPY.Close). The beta is calculated as the coefficient of the market returns term in the model, and this is added to the data frame as a new column using the “mutate()” function.

Asset	**Beta**
AAPL	1.245248
AMRK	0.468529
ATCO	0.905087
FANG	1.315246
NGLOY	1.289607
SHEL	0.94817
TJX	0.998503

Table 7. Betas of stocks

If the beta is 1, then the asset will move along with the market, while a beta less than 1 indicates it will be less volatile than the market, and a beta greater than 1 indicates it will be more volatile than the market. From the beta of our portfolio, we can see that it is about 22% more volatile than the market.

	W1	W2	W3	W4	W5
Beta of the portfolio	1.222408	1.146855	1.322274	1.195695	1.242351
Beta bull	1.197267	1.117823	1.283019	1.166580	1.218263
Beta bear	1.195798	1.164688	1.248920	1.189823	1.201821

Table 8. Beta of the portfolios

Beta is a key concept in finance because it allows investors to compare an asset's risk with that of other assets. A financial institution can also use it to manage their investment portfolios' risks.

Calculating the VaR

In its most general form, the Value at Risk measures the potential loss in value of a risky asset or portfolio over a defined period for a given confidence interval [6]. It represents the maximum loss that an investor is willing to accept on the portfolio or position. VaR is typically used by financial institutions, such as banks and hedge funds, to manage and reduce the risk of their investment portfolios. We will use below formula to calculate the value at risk:

$$VaR = \Delta S(\mu\delta t - \sigma(\delta t)^{\frac{1}{2}}\alpha(1-c)$$

where:
S – Price
Δ – Amount
σ – Volatility
δt – Period
c – Degree of confidence
$\alpha(\cdot)$ – Inverse cdf

In R programming language we can use "VaR()" function form "PerformanceAnalytics" package to calculate the VaR. We will be using the "gaussian" method with 95% confidence level. The "gaussian" method assumes that the portfolio returns follow a normal distribution, which is a common assumption in finance.

From the calculations we have the value of VaR 0.02728567. The VaR result of 0.02728567 is the maximum loss that is expected to be suffered by the portfolio with a 95% probability over the specified time horizon. This means that there is a 5% probability that the portfolio will suffer a loss greater than 2.73% of the total value of the portfolio.

NGLOY, AAPL and AMRK are the three stocks with most contributions. It means those stocks are high-risk investments but with high return. The "$pct_contrib_VaR" is the values of contribution of each asset in percentage and it is provided below:

	VaR (5%)	FANG	ATCO	SHEL	AAPL	NGLOY	AMRK	TJX
W1	0.02729395	5.79%	3.95%	4.10%	26.99%	44.19%	12.22%	2.77%
W2	0.02702680	23.04%	15.64%	14.40%	8.68%	19.43%	9.50%	9.32%
W3	0.04739745	1.33%	1.15%	-2.60%	35.02%	48.79%	12.61%	3.70%
W4	0.02682827	14.48%	9.84%	9.57%	17.35%	33.24%	9.26%	6.26%
W5	0.02892088	0.00%	0.00%	0.00%	34.72%	51.07%	14.21%	0.00%

Table 9. VaR of the portfolios and asset contributions

2.5. ARCH/GARCH volatility estimation

The main objective of any investor is to maximize the expected returns or to minimize the risk subject to some constrains. The GARCH model is fitted to the time series data in order to predict its volatility. This is achieved through the estimation of parameters by the Maximum Likelihood Estimation, et al [7].

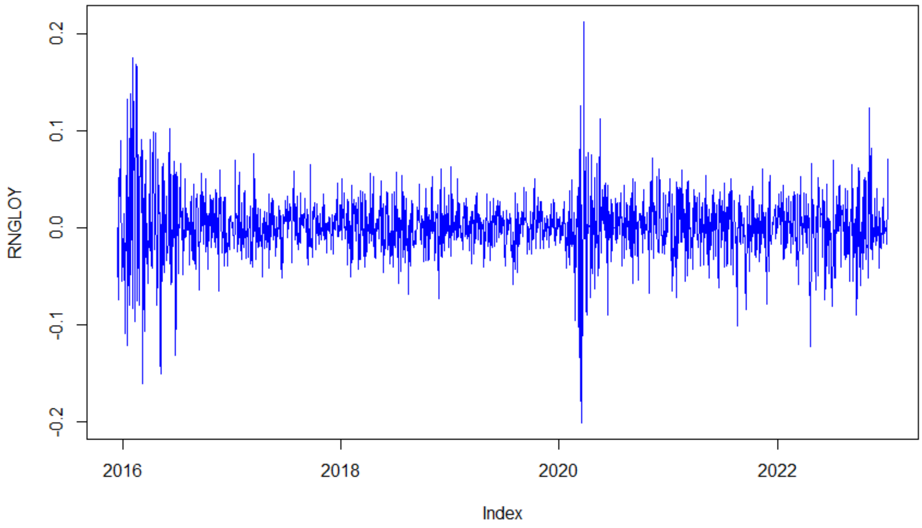

Figure 4. Log returns of NGLOY

We can use the garch() function in R to fit different versions of GARCH (Generalized Autoregressive Conditional Heteroskedasticity) models to a time series of log returns of a NGLOY. The garch() function is used to fit different GARCH models, which are typically used to model the volatility of financial time series data.
Then we can fit different GARCH models, specified by the order of the autoregressive (AR) and moving average (MA) terms, using the garch() function. The trace argument controls whether to print the progress of the estimation process to the console or not, hence for M10, M20, M21, M22, M02, M12 it is false and for M11 it is true.

	df	AIC
M10	2	-7183.34
M11	3	-7766.64
M20	3	-7178.86
M21	4	-7762.71
M22	5	-7714.99
M02	3	-7468.21
M12	4	-7760.76

Table 10. AIC results of all the fitted models

AIC (Akaike Information Criterion) is a measure of the relative quality of a statistical model, with a lower AIC indicating a better model. The best model is the one that has the lowest AIC value among the considered models. In our case the best model is the M11 with the lowest result -7766.64.

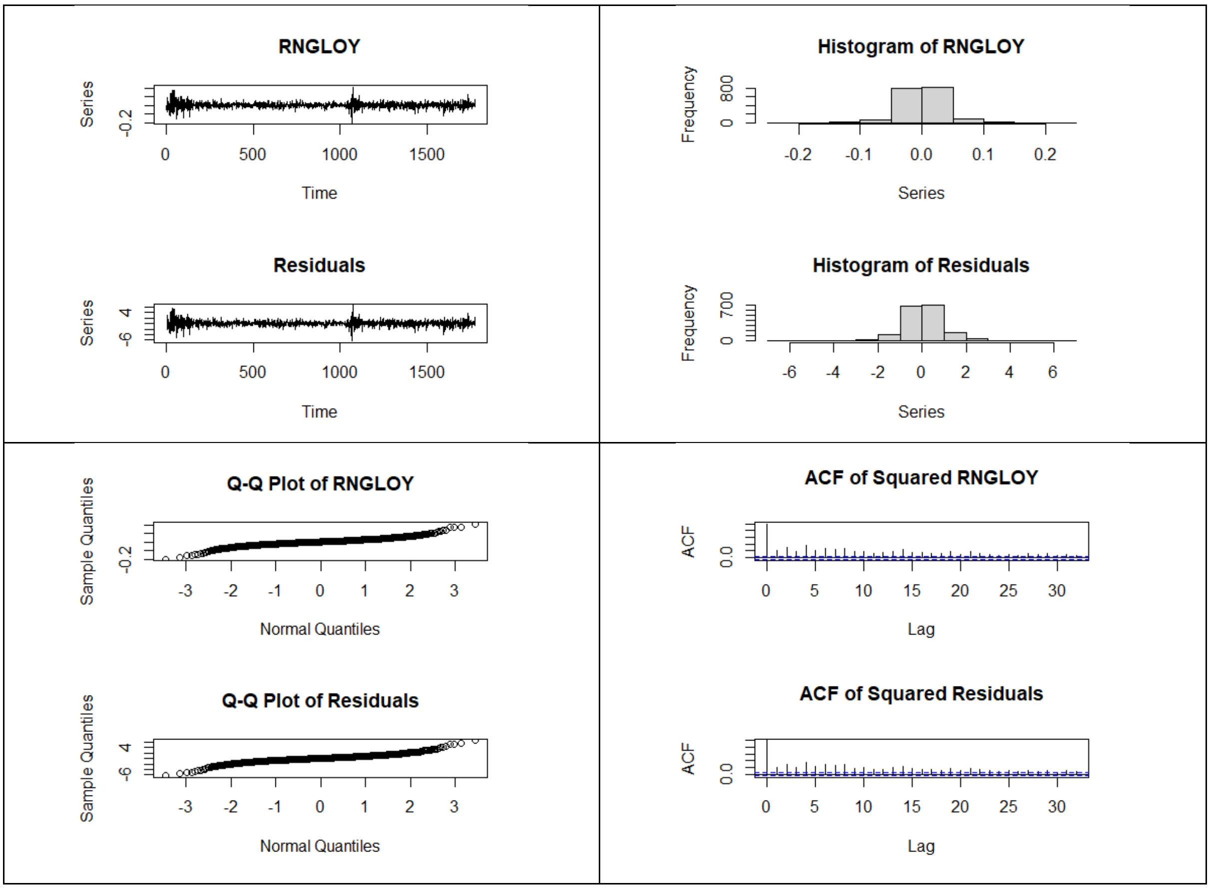

Figure 5. M11 residual plots

From the summary of M11 model we have p-values for Jarque Bera test and Box-Ljung test. The results of tests p-value < 2.2e-16 and p-value = 0.3448 respectively. The p-value for Box-Ljung test is greater than 0.05 that indicates residuals are independent. The p-value for Jarque Bera test is less than 0.05 and it is very small, which might indicate that the model is not best fit. Small value for Jarque Bera test indicates that the residuals are not normally distributed, which suggests that the GARCH model is not a good fit for the data even though it is the best model according to AIC results. To get the p-value we need we can transform the data, change the distribution assumption or add additional explanatory variables.

Residuals:

Min	1Q	Median	3Q	Max
-5.58965	-0.53379	0.05254	0.62762	3.68999

Table 11. Residuals of M11

Coefficient(s):

	Estimate	Std. Error	t value	Pr(>|t|)
a0	1.52E-05	3.77E-06	4.038	5.39e-05 ***
a1	7.26E-02	8.77E-03	8.275	2.22e-16 ***
b1	9.09E-01	1.07E-02	84.603	< 2e-16 ***

Table 12. Coefficients of M11

3. Conclusion

The portfolio has been optimized by determining the portfolio risk and the percentage of investment in each asset in the portfolio for a chosen target return and drawing the efficient frontier curve. The best portfolio has an annualized return of 0.1911 and an annualized standard deviation of 0.3226. The portfolio has an annualized Sharpe ratio of 0.5370 with 1.51% risk-free rate of return.

All the portfolios have a beta of >1, which means that the portfolio's returns are expected to be more volatile than the market returns. The betas of individual assets have been also calculated, and most of the assets have betas equal to or greater than 1, indicating that the assets have returns that are more volatile than the market returns.

The portfolios have an average Value at Risk (VaR) of 0.027, this means that there is a 5% probability that the portfolio will suffer a loss greater than 2.7% of the total value of the portfolio.

In conclusion, from above calculations we can see that the portfolios have low return with higher volatility. Most of the stocks are more volatile than the market and have less returns. Considering this it would be better option to do more research on other stocks with low volatility and diversify the portfolio with the negative correlated assets. Another way is to use some portfolio optimization techniques like Markowitz optimization or Black-Litterman model. These techniques allow to optimize the portfolio based on the expected returns, variances and covariances of the assets, and also takes into account the investor's risk preferences.

It's also important thing to keep in mind that the stock market is constantly changing, so it's essential to regularly review and rebalance the portfolio to ensure that it continues to meet the investor's goals and risk tolerance.

4. References

[1] Markus K. Brunnermeier. Lecture Notes for the course FIN 501: *Asset pricing I: Pricing Models*, 2015.

[2] Hudson, Robert and Gregoriou, Andros, *Calculating and Comparing Security Returns is Harder than you Think: A Comparison between Logarithmic and Simple Returns*, 2010.

[3] Adam Hayes. (2022). *Correlation: What It Means in Finance and the Formula for Calculating It* [Online]. Available: https://www.investopedia.com/terms/c/correlation.asp

[4] Will Kenton. (2022). *Beta: Definition, Calculation, and Explanation for Investors* [Online]. Available: https://www.investopedia.com/terms/b/beta.asp#:~:text=Beta%20(%CE%B2)%20is%20a%20measure,volatile%20than%20the%20S%26P%20500.

[5] Akhilesh Ganti. (2022). *Understanding Capital Market Line (CML) and How to Calculate It* [Online]. Available: https://www.investopedia.com/terms/c/cml.asp

[6] Aswath Damodaran, "VAR" Accessed: Apr. 18, 2007. [Online]. Available: https://pages.stern.nyu.edu/~adamodar/pdfiles/papers/VAR.pdf

[7] Koima, J.K, Mwita, P.N Nassiuma, D.K. "Volatility Estimation of Stock Prices using Garch Method", *European Journal of Business and Management*, Vol.7, No.19, 2015.

[8] Paul Wilmott, *Quantitative Finance*, *2nd Edition.* John Wiley & Sons Ltd, 2007.

[9] Dr Jonathan Iworiso, Dr Abu Hossain. Lecture Notes and Workshop Tasks for the course MA7008: *Financial Mathematics*, 2022.